Lecture 15

Monday, November 4, 2019 6:15 AM

<u>Recall</u>. A Möbius transformation (MT) is a map $S: \mathbb{C}_\infty \to \mathbb{C}_\infty$ of form

$$S(z) = \frac{az+b}{cz+d} \; ; \; ad-bc \neq 0.$$

- Finish material from Lecture Notes 13-14.

<u>Thm1</u>. An MT maps lines/circles to lines/circles.

<u>Pf.</u> (1) Let $S(z) = \frac{az+b}{cz+d}$. First show $S^{-1}(\mathbb{R})$ is a line/circle.

$$z \in S^{-1}(\mathbb{R}) \iff \frac{az+b}{cz+d} = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}} \Rightarrow \text{cross multiply} \dots \Rightarrow$$

$$(a\bar{c}-\bar{a}c)|z|^2 + (a\bar{d}-\bar{b}c)z - (\bar{a}d-b\bar{c})\bar{z} + (b\bar{d}-\bar{b}d) = 0$$

- $a\bar{c}-\bar{a}c \neq 0 \Rightarrow |z|^2 + \bar{\alpha}z + \alpha\bar{z} + \beta = 0$, $\beta \in \mathbb{R}$

  Consider $|z+\alpha|^2 = (z+\alpha)(\bar{z}+\bar{\alpha}) = |z|^2 + \bar{\alpha}z + \alpha\bar{z} + |\alpha|^2 \Rightarrow$

  $|z|^2 + \bar{\alpha}z + \alpha\bar{z} + \beta = |z+\alpha|^2 + \beta - |\alpha|^2 = 0$, i.e. $S^{-1}(\mathbb{R})$ is given by equation $|z+\alpha|^2 = |\alpha|^2 - \beta$ which is a circle if $|\alpha|^2 - \beta > 0$, pt if $|\alpha|^2 = \beta$, $\emptyset$ if $\beta > |\alpha|^2$.

  But we know S is homeo $\mathbb{C}_\infty \to \mathbb{C}_\infty \Rightarrow S^{-1}(\mathbb{R})$ cannot be $\emptyset$ or pt. $\Rightarrow S^{-1}(\mathbb{R})$ is circle.

- $a\bar{c}-\bar{a}c = 0 \Rightarrow \bar{\alpha}z - \alpha\bar{z} + 2i\gamma = 0$, $\gamma \in \mathbb{R}$

  $\Rightarrow S^{-1}(\mathbb{R})$ is given by $\operatorname{Im}(\bar{\alpha}z + \gamma) = 0$; line.

Thus, for any S, $S^{-1}(\mathbb{R})$ is line/circle.

(2) Let S be MT, and Γ any circle/line in $\mathbb{C}$. Γ is uniquely determined by 3 pts on it. Pick $z_1, z_2, z_3 \in \Gamma$. Let $w_j = S(z_j)$, $j = 1,2,3$, and let

$z_1, z_2, z_3 \in \Gamma$. Let $w_j = S(z_j)$, $j = 1,2,3$, and let T_1, T_2 be MT taking z_1, z_2, z_3 and w_1, w_2, w_3 to $1, 0, \infty$ respectively. Then, $S = T_2^{-1} \circ T_1$.

Now, by ① $T_1^{-1}(\mathbb{R})$ is a line or circle w/ z_1, z_2, z_3 on it, hence, $T_1^{-1}(\mathbb{R}) = \Gamma$. $T_2^{-1}(\mathbb{R})$ is a line/circle Γ' w/ w_1, w_2, w_3 on it. Now, $S(\Gamma) = T_2^{-1}(T_1(\Gamma))$

$= T_2^{-1}(\mathbb{R}) = \Gamma'$. ∎

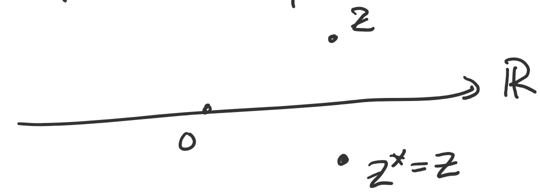
<u>Remark.</u> We think of a line as a circle (w/ $r = \infty$) on $\mathbb{C}_\infty$ through $z = \infty$.

<u>Thm 2.</u> For every pair of circles/lines (Γ, Γ') $\exists$ (not unique!) MT S s.t. $S(\Gamma) = \Gamma'$.

<u>Pf.</u> Take 3 pts on each circle and let S be MT sending one triplet to the other ∎

<u>Symmetric points.</u>

For $z \in \mathbb{C}$, it is natural to define the $\overbrace{\text{symmetric}}^{\text{(mirror)}}$ pt $z^* \in \mathbb{C}$ wrt $\mathbb{R}$ to be $z^* = \overline{z}$

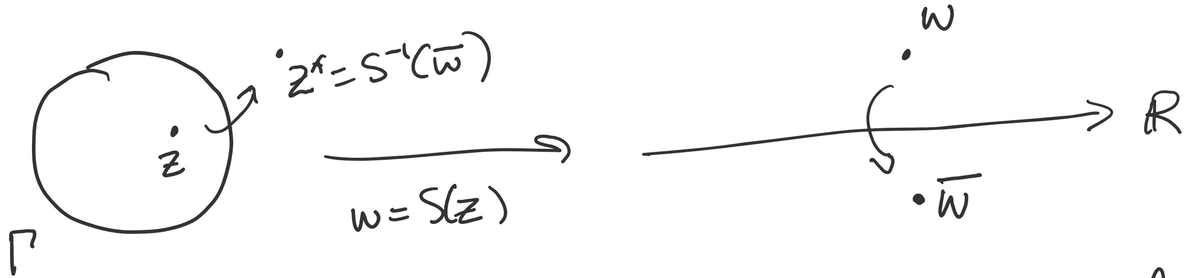


<u>Def.</u> Let line/circle Γ and $z \in \mathbb{C}_\infty$, we define the <u>symmetric point</u> z^* to be as follows. Pick $z_1, z_2, z_3 \in \Gamma$, let S send $(z_1, z_2, z_3) \to (1, 0, \infty)$ and

Def. $z^* = S^{-1}(\overline{S(z)})$

$z_1, z_2, z_3 \in \Gamma$, let S send $(z_1, z_2, \ldots$

let $z^* = S^{-1}(\overline{S(z)})$.



One must of course check that the def only depends on Γ and <u>not</u> on choice (z_1, z_2, z_3). Left as <u>Ex</u>: Consider $T = S_1 \circ S_2^{-1}$, corresponding to 2 different triplets on Γ. $T(\mathbb{R}) = \mathbb{R}$, show $\Rightarrow$ $T(\overline{w}) = \overline{T(w)}$ for all w.